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Algorithm for Solving Cubic Polynomial Spline Interpolation Two Point Boundary Value Problems

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ABSTRACT- In this research paper we discuss the algorithm for solving cubic polynomial Spline Interpolation two point boundary value problems for Bickley's method and numerical procedure for getting the approximate solution. If $\left(s_m\ ,\mathrm{C}^{\mathrm{k}}\right)$ denote the class of spline functions consisting of piecewise cubic polynomial, joined at "knots" with at-least k continuous derivatives $(k<m)$. Spline function in $\left(s_m\ ,\mathrm{C}^{\mathrm{m-1}}\right)$ are used as approximating function for two point boundary value problem in the second order differential equations. The result shows that the presented cubic spline function which interpolates the lacunary data was efficient and effective for solving such problem.

KEYWORDS- Spline Function, Bickley's Method, Lacunary Interpolation
AMS Subject classifications number: 65L10

I. INTRODUCTION

There are some popular methods for equally spaced value of independent variables. These discrete variable methods furnish the approximate solution of differential equations For two point boundary value problem, W Bickley derived a better approximate solution experimentally with the help of a cubic spine function. [2].

The objective of research paper is the thorough studies of theoretical and practical aspects of Bickley's method and use these aspects to obtain the approximate solution of two point boundary value problems.

To advance we require the given consistency relationship for all spline functions in $\left(s_m\ ,\mathrm{C}^{\mathrm{m-1}}\right)$ with uniform knots i.e. $x=\nu h,\ \nu$=1, 2, 3, 4........., (n-1), which are generalization of theorem in [3].

Theorem 1: For any spline function $s(x)\in\left(S_m,\ \mathrm{C}^{\mathrm{m-1}}\right),\ \mathrm{m}\geq 3,$ there are linear relations Between the quantities $s(\nu h),\ \mathrm{s}'(\nu h);\ \mathrm{s}(\nu h),\ \mathrm{s}''(\nu h),\ \nu$=0,1,2,......(m-1), given by:-

(1.1)

$$\sum_{\nu=0}^{m-1} a_{\nu}^{(m)}\ s(\nu h)\ =\ h\sum_{\nu=0}^{m-1} b_{\nu}^{(m)}\ s'(\nu h),$$

(1.2)

$$\sum_{\nu=0}^{m-1} \mathrm{c}_{\nu}^{(m)}\ s(\nu h)\ =\ h^2\sum_{\nu=0}^{m-1} b_{\nu}^{(m)}\ s''(\nu h),$$

Whose coefficients are

(1.3)

$$a_{\nu}^{(m)}=(\mathrm{m\text{-}1})!\ \left(\mathrm{Q_m}(\nu)-Q_m(\nu+1)\right),$$

(1.4)

$$c_{\nu}^{(\mathrm{m})}=(\mathrm{m\text{-}1})!\left(\mathrm{Q_{m\text{-}1}}(\nu+1)-2Q_{m-1}(\nu)+Q_{m-1}(\nu-1)\right),$$

(1.5)

$$b_{\nu}^{(\mathrm{m})}=(\mathrm{m\text{-}1})!\ \mathrm{Q_{m+1}}(\nu+1),$$

Where,

$Q_{m+1}(x)=\dfrac{1}{m!}\displaystyle\sum_{i=0}^{m+1}(-1)^i\binom{m+1}{i}(x-i)_+^m$ is a B-spline.

***Theorem 2*:** Under the same condition of the theorem 1, there are the following reations:

(1.6)

$$\sum_{\nu=-1}^{m-1}\bar{\mathrm{a}}_{\nu}^{(m)}\ s\left(\left(\nu+\frac{1}{2}\right)h\right)\ =\ h\sum_{\nu=-1}^{m-1}\bar{\mathrm{b}}_{\nu}^{(m)}\ s'\left(\left(\nu+\frac{1}{2}\right)h\right),$$

(1.7)

$$\sum_{\nu=-1}^{m-1}\bar{\mathrm{c}}_{\nu}^{(m)}\ s\left(\left(\nu+\frac{1}{2}\right)h\right)\ =\ h^2\sum_{\nu=-1}^{m-1}\bar{\mathrm{b}}_{\nu}^{(m)}\ s''\left(\left(\nu+\frac{1}{2}\right)h\right),$$

Whose coefficient are

(1.8)

$$\bar{a}_{\nu}^{(m)} = (\text{m-1})!\left(Q_m\left(\nu+\frac{1}{2}\right)-Q_m\left(\nu+\frac{3}{2}\right)\right),$$

(1.9)

$$\bar{c}_{\nu}^{(m)} = (\text{m-1})!\left(Q_{m-1}\left(\nu+\frac{3}{2}\right)-2Q_{m-1}\left(\nu+\frac{1}{2}\right)+Q_{m-1}\left(\nu-\frac{1}{2}\right)\right),$$

(1.10)

$$\bar{b}_{\nu}^{(m)} = (\text{m-1})!\, Q_{m+1}\left(\nu+\frac{3}{2}\right).$$

We will consider the existence and uniqueness theorem of the approximate solution and check the convergence when $m=3$ for the differential equation:

$$-y''(x)+\sigma(x)(x) == f(x), \quad \sigma(x)\geq 0,$$

with boundary conditions

$$y(0)=\alpha, \quad y(1)=\beta$$

II. CUBIC SPLINE INTERPOLATION

A cubic spine function $w(x)$ in interval $\left[x_0, x_1\right]$, starting with point x_0 can be given as:

(2.1)

$$w(x)=a+b(x-x_0)+c(x-x_0)^2+d_0(x-x_0)^3$$

As we proceed to next interval $\left[x_1, x_2\right]$, we add a term $d_1(x-x_1)^3$ in the equation (2.1). Simmilarly for interval $\left[x_2, x_3\right]$,we add $d_2(x-x_2)^3$, for interval $\left[x_3, x_4\right]$,we add $d_3(x-x_3)^3$,........................... for interval $\left[x_{n-1}, x_n\right]$,we add $d_n(x-x_n)^3$, therefore spline function for intervals can be defined as:

(2.2)

$$w(x)=a+b(x-x_0)+c(x-x_0)^2+\sum_{i=0}^{n-1}d_i(x-x_i)_+^3$$

where,

$$x_+^m=\begin{cases} x^m & if \quad x>0 \\ 0 & if \quad x\leq 0 \end{cases}$$

The number of coefficient in equation (2.2) is (n+3).
The boundary conditions give us two equations towards the determination of these. The remaining (n+1) parameters are to be determined with the help of (n+1) nodes appeared in function. The satisfaction of the differential equation by these (n+1) nodes provides the required numbers of equations.
The equation to be satisfied at the i^{th} node is:-

(2.3)

$$-w_i''+\sigma_i w_i = f_i \qquad 0\leq \text{i} \leq \text{n}.$$

In above equation we also add those parameter obtained by satisfying boundary conditions, namely,

(2.4)

$$w_0=\alpha, \qquad w_n=\beta.$$

For cubic spline function $w(x)$ there are the following relationships from theorem 1.

(2.5)

$$w_{i+1}-2w_i+w_{i-1}=\frac{h^2}{6}\left(w_{i+1}''+4w_i''+w_{i-1}''\right), \quad 1\leq \text{i} \leq \text{n-1}$$

(2.6)

$$w_{i+1}-w_{i-1}=\frac{h}{3}\left(w_{i+1}'+4w_i'+w_{i-1}'\right), \quad 1\leq \text{i} \leq \text{n-1}$$

We can also see a relationship between

$$w_\nu' \text{ and } w_\nu'' \quad , \ \nu = i, \text{i+1}$$

(2.7)

$$w_{i+1}'-w_i'=\frac{h}{2}\left(w_{i+1}''+w_i''\right),$$

From (2.3) and (2.5), we obtain the reation

(2.8)

$$\left(-1+h^2\frac{\sigma_{i+1}}{6}\right)w_{i+1}+\left(2+\frac{2}{3}h^2\sigma_i\right)w_i+\left(-1+h^2\frac{\sigma_{i-1}}{6}\right)w_{i-1}$$

$$=\frac{h^2}{6}\left(f_{i+1}+4f_i+f_{i-1}\right), \ 1\leq \text{i} \leq \text{n-1}$$

If we write $W=\left(w_1, w_2, w_3.........w_{n-1}\right)^T$, then the set of equations given by (2.8) may be written in the form

(2.9) $$AW=K$$

Where K is a vector with (n-1) components,

$$k_1=\frac{1}{6}\left(f_0+4f_1+f_2\right)-\frac{\alpha}{h^2}\left(-1+\frac{\sigma_0}{6}h^2\right),$$

$$k_i=\frac{1}{6}\left(f_{i-1}+4f_i+f_{i+1}\right), \quad 2\leq \text{i} \leq \text{n-2}$$

$$k_{n-1}=\frac{1}{6}\left(f_{n-2}+4f_{n-1}+f_n\right)-\frac{\beta}{h^2}\left(-1+\frac{\sigma_n}{6}h^2\right),$$

And the matrix A is a tridiagonal matrix of order (n-1)

$$a_{i,\text{i-1}}=\frac{1}{h^2}\left(-1+h^2\frac{\sigma_{i-1}}{6}\right), \quad a_{i,\text{i}}=\frac{1}{h^2}\left(2+\frac{2h^2}{3}\sigma_i\right),$$

$$a_{i,\text{i+1}}=\frac{1}{h^2}\left(-1+\frac{h^2}{6}\sigma_{i+1}\right)$$

As A is a tridiagonal M-matrix $([4:p.85])$, where $0< h \le \sqrt{\dfrac{6}{\max\sigma(x)}}$, this matrix problem can be directly solved by a simple algorithm.
From (2.3), (2.6) and (2.8), we get the relations:-

(2.10)

$$w_0' = w_1' - \frac{h}{2}(\sigma_1 w_1 + \sigma_0 w_0 - f_0 - f_1), \text{ n}$$

(2.11)

$$w_i' = \frac{1}{2h}(w_{i+1} - w_{i-1}) - \frac{h}{12}(\sigma_{i+1}w_{i+1} - \sigma_{i-1}w_{i-1} - f_{i+1} + f_{i-1}), \quad 1 \le i \le n-1$$

(2.12)

$$w_n' = w_{n-1}' + \frac{h}{2}(\sigma_n w_n + \sigma_{n-1}w_{n-1} - f_n - f_{n-1}).$$

From (2.10), (2.11) and (2.12), the derivatives at nodes can be calculated, and w(x) can be obtained by Hermite interpolation by a polynomial of degree 3 on each interval.

Theorem 3. If the solution y(x) of the equation is in $C^4[0,\ 1]$, then we have the following constants K_1 and K_2 such that

(2.13

$$|y(x) - w(x)| \le K_1 h^2, \qquad |y'(x) - w'(x)| \le K_2 h.$$

Proof: By Taylor expansions and Gerschgorin type convergence arguments, we can easily obtain

(2.14) $$|y_i - w_i| \le Kh^2.$$

From (2.10), (2.11) and (2.12), we get

(2.15) $$|y_i' - w_i'| \le Kh.$$

To complete the proof of above theorem we need one more important result from the following theorem:-

Theorem 4. Let $\Upsilon(x) \in C^t[0,1]$ *with* $t \ge 2m$, and w(x) be in $(S_{2m-1},\ C^{m-1})$ such that for some p, $(2m-2 \le p \le 2m)$,

$$|D^t((x_i) - w(x_i))| \le Mh^{p-t} \quad \forall\ 0 \le i \le n,\ 0 \le t \le m-1.$$

If $x \in [0,\ 1]$, then

$$|D^k(\Upsilon(x) - w(x))| \le Mh^{p-k} \quad \forall k \ \text{ with } 0 \le k \le m-1.$$

Proof: Let us consider the polynomial $p(x) \in (S_{2m-1},\ C^{m-1})$ which interpolates the data, $\Upsilon^{(\nu)}(x_k),\ \Upsilon^{(\nu)}(x_{k+1}),\ \nu = 0,1,2,........(m-1).$ then there exists a constant M_1 such that:-

(2.16) $$|D^k(\Upsilon(x) - p(x))| \le M_1 h^{2m-k} \quad \forall k \ \text{ with } 0 \le k \le m-1.$$

For estimating $p(x) - w(x)$ in each interval $I_k = [kh,\ (k+1)h]$, we use Hermite interpolation formula

(2.17)

$$p(x) - w(x) = \sum_{j=0}^{m-1} h^j \left\{ \begin{array}{l} \left(p^{(j)}(kh) - w^{(j)}(hk)\right)\lambda_j\left(\dfrac{x-kh}{h}\right) \\ +(-1)^j\left(p^{(j)}((k+1)h) - w^{(j)}((k+1)h)\right)\lambda_j\left(k+1-\dfrac{x}{h}\right) \end{array} \right\},$$

Where $\lambda_j(t)$ are polynomials of degree 2m-1 which are the fundamental functions.
From the assumption of the theorem we have,

$$\left|p^{(j)}(kh) - w^{(j)}(hk)\right|, \left|p^{(j)}((k+1)h) - w^{(j)}((k+1)h)\right| \le Mh^{p-j}.$$

Therefore we have:

(2.18)

$$\left|D^k\left(p(x) - w(x)\right)\right| \le M_2 h^{p-k} \quad \forall k \ \text{ with } 0 \le k \le m-1.$$

On combining (2.16) and (2.18), we get the desired result.

Algorithm for C^3 – Cubic Spline function:
The algorithm of cubic spline function can be discussed under following steps:

Step 1: Divide the interval [0, 1] into n small subintervals.

Step 2: We evaluate $S_0''' = y_0'''$ for given value $s(0) = y(0),\quad s'(0) = y'(0) \text{ and } s''(0) = y''(0)$

Step 3: $S_i'' = f(x_i, s_i)$ for i = 1, 2, 3, ...n, $x \in [x_{i-1}, x_i]$.

Step 4: Evaluate the value of s_i and s_i' for (n+1) equally spaced value , with the help of Taylor series expansion as:

$$s_i = s_{i-1} + hs_i' - \frac{h^2}{20}\left[3f(x_{i-1}, s_{i-1}) + 7f(x_i, s_i)\right] + \frac{h^3}{60}s''' \ and$$

$$s_i' = s_{i-1}' + \frac{h}{2}\left[3f(x_{i-1}, s_{i-1}) + f(x_i, s_i)\right]$$

Step 5: Evaluate:-

$$s_{i+1}''' = -5s_i''' + \frac{3}{h}\left[f(x_{i+1}, s_{i+1}) - f(x_{i-1}, s_{i-1})\right]$$

Step 6: If $i = k+1$, go to next step , otherwise repeat the process from step 4.

Step 7: Stop the process.

Problem: Let us consider the problem:

$$y'' = x + y, \quad y(0)=1,\ y'(0)=0, \quad x \in [0,\ 1]$$

Results: Absolute maximum error for s(x) and $s'(x)$ with several values of 'h'

h	e	e'	e''
$\frac{1}{10}$	3.14×10^{-6}	8.36×10^{-5}	1.27×10^{-4}
$\frac{1}{20}$	1.01×10^{-5}	4.97×10^{-4}	2.97×10^{-7}

III. CONCLUSION:

To formulate the second order initial value problems, I have used cubic C^3-Lacunary spline function in my work. The algorithm used in formulation of problem, gives better solution for second order initial value problems. In the formulation I have used small strips (h) to formulate and test the results. It is matter of research or experiments that the above algorithm will also provides the solution of higher order initial value problems.

CONFLICTS OF INTEREST

The authors declare that they have no conflicts of interest

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