Random Euler Method for Solving a System of Random Matrix Differential Initial Value Problem in Mean Square Sense

M. A. Sohaly, Ahlam H.Tolba

Abstract---**In this paper, the random Euler method (REM) is used in solving system of random matrix differential initial value problems of first order. The existence and uniqueness theorem was proved. The REM is presented and the conditions for the mean square (m.s.) convergence are established. Numerical examples show that REM gives good results where some statistical properties of the numerical solutions are computed through numerical case studies.**

Keywords: **Random Differential Equations(RDE's), Mean Square Sense(m.s.), Matrix initial Value Problems, Random Euler Method(REM).**

I. INTRODUCTION

Random differential equations (RDE's) are useful to model problems involving rates of changes of quantities representing variables under uncertainties or randomness, being in fact stochastic processes instead of deterministic functions [1], [2], [3], [4], [5] and [6]. The concept of mean square calculus is applied to study system of RDE's. The REM is used to obtain an approximate solution for equation (1.1).

$$\left.\begin{aligned}\dot{\boldsymbol{X}}(t) &= \boldsymbol{F}(\boldsymbol{X}(t),t), \qquad t \in T\\ \boldsymbol{X}(t_0) &= \boldsymbol{X}_0\end{aligned}\right\} \qquad (1.1)$$

where $\boldsymbol{X}_0$ is a second order random matrix of size $r \times s$ and, the unknown $\boldsymbol{X}(t)$ *is* a second order random matrix (2-r.m.) of size $r \times s$ and $\boldsymbol{F}(\boldsymbol{X}(t),t)$ are matrix stochastic processes of size $r \times s$, $T = [t_0, t_e]$.

This paper is organized as follows. In section 2, we deal with some preliminary definitions, results, notations and examples. Section 3, is addressed to the proof of existence and uniqueness theorem. Section 4, is addressed to the presentation and the proof of the convergence for the random Euler Scheme in mean square sense. In Section 5, some statistical properties for the exact and numerical solutions are studied. The last section is devoted to conclusions.

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II. PRELIMINARIES

Definition2.1. [7]

If $\{X^{ij}: 1 \le i \le r, 1 \le j \le s\}$ is the set of $r \times s$ 2-r.v.'s, then the second order random matrix (2-r.m.) associated to this family is defined as

$$\boldsymbol{X} = \begin{bmatrix} X^{11} & \cdots & X^{1s} \\ \vdots & \ddots & \vdots \\ X^{r1} & \cdots & X^{rs} \end{bmatrix} \qquad (1.2)$$

The set of all 2-r.m.'s $\boldsymbol{X}$ of size $r \times s$ endowed with the norm

$$\|\boldsymbol{X}\|_{r\times s} = \max_{1\le i\le r} \sum_{j=1}^{s} \left\|X^{ij}\right\|, \quad (1.3)$$

Has a Banach space structure, denoted by $L_2^{r\times s}$. Where using the notation introduced at (1.2),

$$E[\boldsymbol{X}] = (E[X^{ij}])_{r\times s}.$$

The m.s. continuity, m.s. differentiation, and m.s. integration associated with a second –order matrix stochastic process are defined with respect to the norm $\|\boldsymbol{X}\|_{r\times s}$.

Definition2.2. [7]

A sequence of 2-r.m.'s $\{\boldsymbol{X}_n\}_{n\ge 0}$ is mean square (m.s.) convergent to $\boldsymbol{X} \in L_2^{r\times s}$, and will be denoted by $\boldsymbol{X}_n \xrightarrow{m.s.} \boldsymbol{X}$, as: $n \to \infty$,

if: $\lim_{n\to\infty} \|\boldsymbol{X}_n - \boldsymbol{X}\|_{r\times s} = 0. \quad (1.4)$

Theorem2.1. [7]

if $\{\boldsymbol{X}_n\}_{n\ge 0}$ is a sequence of random matrices in $L_2^{r\times s}$ m.s. convergent to $\boldsymbol{X}$, then:

$$E[\boldsymbol{X}_n] \xrightarrow[n\to\infty]{} E[\boldsymbol{X}], \qquad (1.5)$$

Difintion2.3. [7]

A 2-m.s.p.$\{\boldsymbol{X}(t): t \in T\}$ *in* $L_2^{r\times s}$ is m.s. continuous at $t \in T$, T an interval of real line, if

$$\lim_{\tau\to 0} \|\boldsymbol{X}(t+\tau) - \boldsymbol{X}(t)\|_{r\times s} = 0, \; t, t+\tau \in T$$

(1.6) Difintion2.4.[7]

A 2-m.s.p. $\{\boldsymbol{X}(t): t \in T\}$ *in* L_2^{r} is m.s. differentiable at $t \in T$, if there exists a 2-m.s.p. denoted by $\{\dot{X}(t): t \in T\}$ such that:

$$\lim_{\tau\to 0} \left\| \frac{\boldsymbol{X}(t+\tau) - \boldsymbol{X}(t)}{\tau} - \dot{\boldsymbol{X}}(t) \right\|_{r\times s} = 0, \; t, t+\tau \in T \quad \textbf{(1.7)}$$

Example2.1.[8]

Let Y be a 2-r.m. and let us consider the 2-s.p. $Y(t) = Y.t$ for t lying in the interval T and applying the formula of integration by parts for $h(t,u) \equiv 1$ one gets:

$$\int_{t_0}^{t} Y\, du = (t - t_0).$$

Proposition2.1. [8]
If $\{X(t), t \in T\}$is a 2-s.p. m.s. Continuous on $T = [t_0, t]$, then:

$$\left\| \int_{t_0}^{t} X(u)du \right\| \le \int_{t_0}^{t} \|X(u)\| du \ \le \ M_X(t - t_0),$$

$$, \quad M_X = \max_{t_0 \le u \le t} \|X(u)\|. \qquad (1.8)$$

Definition2.5.[7]
Consider equation (1.1) where:

$$\boldsymbol{F}: S \times T \to L_2^{r\times s} \, with \, S \subset L_2^{r\times s}$$

is continuously and $\boldsymbol{X}_0 \in L_2^{r\times s}$. The m.s.p. $\boldsymbol{X}(t): T \to L_2^{r\times s}$ is called a mean square (m.s.) solution of (3.1) on $S \times T$ if :

1. $\boldsymbol{X}(t)$ is m.s. continuous on T,
2. $\boldsymbol{X}(t_0) = \boldsymbol{X}_0$,
3. $\boldsymbol{F}(\boldsymbol{X}(t), t)$ is the m.s. derivative of $\boldsymbol{X}(t)$ on T.

Theorem 2.2.
$\boldsymbol{X}(t) : T \to L_2^{r\times s}$ is a m.s. solution of (1.1) if and only if, for all t $\in T$,

$$\boldsymbol{X}(t) = \boldsymbol{X}_0 + \int_{t_0}^{t} \boldsymbol{F}(\boldsymbol{X}(s), s) \ ds \qquad (1.9)$$

where the integral is understood to be **m.s. integral.**

III. THE EXISTENCE AND UNIQUENESS

Theorem 3.1.
In equation (1.1), If $\boldsymbol{F}: S \times T \to L_2^{r\times s}$ satisfies the m.s. Lipschitz condition:

$$\|\boldsymbol{F}(\boldsymbol{X}, t) - \boldsymbol{F}(\boldsymbol{Y}, t)\|_{r\times s} \le k(t) \|\boldsymbol{X} - \boldsymbol{Y}\|_{r\times s} \quad (1.10)$$

$$where \qquad \int_{t_0}^{t_e} k(t)dt < \infty \ ,$$

then there exists a unique m.s. solution for any initial condition $\boldsymbol{X}_0 \in L_2^{r\times s}$.

Proof
The existence can be proved by using successive approximations. Let:

$$\boldsymbol{X}_t^0 = \boldsymbol{X}_0 \qquad (1.11)$$

for $n > 1$

$$\boldsymbol{X}_t^n = \boldsymbol{X}_0 + \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{X}_s^{n-1}, s) ds \qquad (1.12)$$

And for $n = 1$ we have:

$$\|\boldsymbol{X}_t^1 - \boldsymbol{X}_t^0\|_{r\times s} = \left\| \int_{t_0}^{t_e} \boldsymbol{F}(\boldsymbol{X}_0, s) ds \right\|_{r\times s} \le c. |t_e t_0|$$

where $\|\boldsymbol{F}(\boldsymbol{X}_0, t)\|_{r\times s} \le c$. For $n > 1$ we have:

$$\|\boldsymbol{X}_t^n - \boldsymbol{X}_t^{n-1}\|_{r\times s} = \left\| \int_{t_0}^{t_e} [\boldsymbol{F}(\boldsymbol{X}_s^{n-1}, s) - \boldsymbol{F}(\boldsymbol{X}_s^{n-2}, s)] ds \right\|_{r\times s}$$

$$\le \int_{t_0}^{t_e} k. \ \|\boldsymbol{X}_s^{n-1} - \boldsymbol{X}_s^{n-2}\|_{r\times s}$$

Successively, we can obtain the following:

$$\|\boldsymbol{X}_t^n - \boldsymbol{X}_t^{n-1}\|_{r\times s} \le \int_{t_0}^{t_e} k. \|\boldsymbol{X}_s^{n-1} - \ \boldsymbol{X}_s^{n-2}\|_{r\times s} ds$$

$$\le \int_{t_0}^{t_e} k. \left[\int_{t_0}^{t_e} k. \|\boldsymbol{X}_s^{n-2} - \ \boldsymbol{X}_s^{n-3}\|_{r\times s} ds \right] ds \ \le \ldots.$$

$$\le k^{n-1}. \int_{t_0}^{t_e} \int_{t_0}^{t_e} \ldots. \int_{t_0}^{t_e} c|s - t_0| \, ds \ldots ds$$

$$\le ck^{n-1} \frac{|t - t_0|^n}{n!} \qquad (1.15)$$

Therefore,

$$\|\boldsymbol{X}_t^n - \boldsymbol{X}_t^0\|_{r\times s} =$$

$$\|(\boldsymbol{X}_t^n - \boldsymbol{X}_t^{n-1}) + (\boldsymbol{X}_t^{n-1} - \boldsymbol{X}_t^{n-2}) + \cdots + (\boldsymbol{X}_t^1 - \boldsymbol{X}_t^0)\|_{r\times s}$$

$$\le \|\boldsymbol{X}_t^n - \boldsymbol{X}_t^{n-1}\|_{r\times s} + \|\boldsymbol{X}_t^{n-1} - \boldsymbol{X}_t^{n-2}\|_{r\times s} + \cdots + \|\boldsymbol{X}_t^1 - \boldsymbol{X}_t^0\|_{r\times s}$$

$$\le c\, k^{n-1} \frac{|t - t_0|^n}{n!} + c\, k^{n-2} \frac{|t - t_0|^{n-1}}{(n-1)!} + \cdots + c \, \frac{|t - t_0|}{1!}$$

$$= \sum_{r=1}^{n} c. k^{(r-1)} \frac{|t - t_0|^{r\times s}}{r!} \qquad (1.16)$$

Note that:

$$\sum_{r=1}^{\infty} ck^{r-1} \frac{|t - t_0|^r}{r!} \, \frac{c}{k} e^{k\|t_e - t_0\|_{r\times s}} \qquad (1.17)$$

is convergent for finite t.
Hence:

$$\lim_{n\to\infty} \|\boldsymbol{X}_t^n - \boldsymbol{X}_t^0\|_{r\times s} \le \frac{c}{k} e^{k\|t_e - t_0\|_{r\times s}}$$

Further,

$$\lim_{n\to\infty} \|\boldsymbol{X}_t^n - \boldsymbol{X}_t^0\|_{r\times s} = \|\boldsymbol{Z}_t\|_{r\times s} \le \frac{c}{k} e^{k\|t_e - t_0\|_{r\times s}} \qquad (1.18)$$

Then $\lim_{n\to\infty} \boldsymbol{X}_t^n$ exists,
i.e . $\quad \boldsymbol{X}_t = \lim_{n\to\infty} \boldsymbol{X}_t^n$.

Then the mean square limit $\mathrm{l.i.m}_{n\to\infty} \boldsymbol{X}_t^n$ exists, since $\boldsymbol{X}_t^n$ is the general solution of equation (1.12) and $\boldsymbol{X}_t$ is the general solution of equation (1.9).

To prove the uniqueness of the solution, let $\boldsymbol{X}_t$ is a solution of the initial value problem (1.1) and $\boldsymbol{Y}_t$ is the solution of

$$\left. \begin{aligned} \dot{\boldsymbol{Y}}(t) = \boldsymbol{F}(\boldsymbol{Y}(t), t), \ t_0 < t < t_e, \\ \boldsymbol{Y}(t_0) = \boldsymbol{Y}_0 \end{aligned} \right\} \qquad (1.19)$$

and for

$$\boldsymbol{Y}(t) = \boldsymbol{Y}_0 + \int_{t_0}^{t_e} F(\boldsymbol{Y}(s), s) ds \qquad (1.20)$$

To complete the proof we need to prove that:

$$\boldsymbol{X}_t = \boldsymbol{Y}_t \ . \qquad (1.21)$$

By subtracting (1.20) from (1.9), we obtain:

$$\boldsymbol{X}_t - \boldsymbol{Y}_t = \boldsymbol{X}_0 - \boldsymbol{Y}_0 + \int_{t_0}^{t} (\boldsymbol{F}(\boldsymbol{X}_s, s) - \boldsymbol{F}(\boldsymbol{Y}_s, s)) ds \qquad (1.22)$$

Since $\boldsymbol{X}_0 = \boldsymbol{Y}_0$, then:

$$\|\boldsymbol{X}_t - \boldsymbol{Y}_t\|_{r\times s} \le \int_{t_0}^{t} k. \|\boldsymbol{X}_s - \boldsymbol{Y}_s\|_{r\times s} ds \qquad (1.23)$$

Or $\qquad \boldsymbol{U}_t \le k. \int_{t_0}^{t} \boldsymbol{U}_s ds \qquad (1.24)$

where $\boldsymbol{U}_t = \|\boldsymbol{X}_t - \boldsymbol{Y}_t\|_{r\times s}$. From equation (1.24) we have:

$$\|\boldsymbol{U}_t\|_{r\times s} \le k. \int_{t_0}^{t} \|\boldsymbol{U}_s\|_{r\times s} ds = k|t - t_0| \|\boldsymbol{U}_t\|_{r\times s} \quad (1.25)$$

at:$t = t_0 \quad then \ \ \|\boldsymbol{U}_t\|_{r\times s} \le 0 \Longrightarrow \|\boldsymbol{U}_t\|_{r\times s} = 0$

From equation (1.25), if $t > t_0$, then k must be satisfied the condition:

$$k \ge \frac{1}{|t - t_0|} \qquad (1.26)$$

The inequality (1.26) is contradiction because k must be an independent free constant, hence the only solution of the integral equation (1.24) is

$$\boldsymbol{U}_t = 0 \ \text{ or } \ \boldsymbol{X}_t = \boldsymbol{Y}_t \qquad (1.27)$$

i.e., the solution of equation (1.1) exists and is unique.

Example 3.1
Consider the linear random vector differential equation

$$\dot{\boldsymbol{X}}(t) = A(t)\boldsymbol{X}(t) - \boldsymbol{Y}(t), \ \ t \ge 0; \quad \boldsymbol{X}(0) = \boldsymbol{X}_0$$

where $A(t) = e^{-t}$ is a deterministic and continuous function, $\boldsymbol{Y}(t)$ is a vector second-order s.p. and $\boldsymbol{X}_0 \in L_2^n$. Show that, by means of theorem 3.1., the mean-square solution of the equation above exists and is unique

Proof
Since $\qquad \boldsymbol{F}(\boldsymbol{X}, t) = A(t)\boldsymbol{X}(t) - \boldsymbol{Y}(t)$
By substituting in (1.3), we get

$$\|\boldsymbol{F}(\boldsymbol{X},t)-\boldsymbol{F}(\boldsymbol{Y},t)\|_n = \|A(t)\boldsymbol{X}(t)-\boldsymbol{Y}(t)-A(t)\boldsymbol{Y}(t)+\boldsymbol{Y}(t)\|_n = \|A(t)\boldsymbol{X}(t)-A(t)\boldsymbol{Y}(t)\|_n \le |A(t)|\|\boldsymbol{X}-\boldsymbol{Y}\|_n$$

Since, $A(t)=e^{-t}$ is a deterministic and continuous function, $t\ge 0$, then:

$$\int_0^\infty A(t)dt=\int_0^\infty e^{-t}\,dt<\infty$$

then $\boldsymbol{F}(\boldsymbol{X},t)=A(t)\boldsymbol{X}(t)-\boldsymbol{Y}(t)$ satisfy the m.s. Lipschitz condition. So the mean-square solution of the equation above exists and is unique.

Definition3.1.[5]

Given a matrix: $\boldsymbol{A}=(a^{ij})\ \ in\ \mathbb{R}^{r\times r}$, we denoted by: $\|\boldsymbol{A}\|_\infty$, the norm defined as:

$$\|\boldsymbol{A}\|_\infty=\max_{1\le i\le r}\sum_{j=1}^{r}|a^{ij}|. \qquad (1.28)$$

The following lemma deals with the norm of the product of the deterministic matrix function by a random matrix of compatible sizes.

Lemma 3.1.[8]

If **A**, **B** be matrices in $\mathbb{R}^{r\times r},\mathbb{R}^{s\times s}$ respectively, and **X**, **Y** $\in L_2^{r\times s}$, then:

$$\|\boldsymbol{A}\,\boldsymbol{X}\|_{r\times s}\le\|\boldsymbol{A}\|_\infty\|\boldsymbol{X}\|_{r\times s}\quad,\|\boldsymbol{Y}\,\boldsymbol{B}\|_{r\times s}\le\|\boldsymbol{B}\|_\infty\|\boldsymbol{Y}\|_{r\times s}. \quad (1.29)$$

Definition 3.2.[8]

Let S be a bounded set in $L_2^{r\times s}$, an interval $T\subseteq\mathbb{R}$ $and\ h>0$, we say that $\boldsymbol{F}: S\times T\to L_2^{r\times s}$ is randomly bounded time uniformly continuous in S if:

$$\lim_{h\to\infty}\omega(S,h)=0, \qquad (1.30)$$

where

$$\omega(S,h)=\sup_{X\in S\subseteq L_2^{r\times s}}\ \sup_{|t-t'|\le h}\|\boldsymbol{F}(\boldsymbol{X},t)-\boldsymbol{F}(\boldsymbol{X},t')\|_{r\times s}. \quad (1.31)$$

IV. CONVERGENCE OF RANDOM EULER METHOD FOR SYSTEM OF RANDOM DIFFERENTIAL EQUATIONS IN MEAN SQUARE SENSE

The corresponding random matrix Euler method associated to (1.1) takes the form:

$$\left.\begin{aligned}\boldsymbol{X}_{n+1}&=\boldsymbol{X}_n+h\boldsymbol{F}(\boldsymbol{X}_n,t_n),\ n\ge 0\\ \boldsymbol{X}_0&=\boldsymbol{X}(t_0),\end{aligned}\right\}\qquad(1.32)$$

where X_n^{ij} is the (i,j)-th entry of the random matrix $\boldsymbol{X}_n$ and $F^{ij}(\boldsymbol{X}_n,t_n)$ is the (i,j)-th entry of $\boldsymbol{F}(\boldsymbol{X}_n,t_n)$. The discretization error $\boldsymbol{e}_n$ has its corresponding (i,j)-th entry

$$e_n^{ij}=X_n^{ij}-X^{ij}(t_n). \qquad (1.33)$$

We assume the same hypotheses C1 and C2 with only difference that the norm in $L_2^{r\times s}$ is $\|\cdot\|_{r\times s}$ introduced in equation (1.3), $\boldsymbol{F}: S\times T\to L_2^{r\times s}$ $with$ $S\subset L_2^{r\times s}$, satisfies:

$-\,\boldsymbol{C1}$: $\boldsymbol{F}(\boldsymbol{X},t)$ Is m.s. randomly bounded time uniformly continuous.

$-\,\boldsymbol{C2}$: $\boldsymbol{F}(\boldsymbol{X},t)$ Satisfies the m.s. Lipschitz condition: $\|\boldsymbol{F}(\boldsymbol{X},t)-\boldsymbol{F}(\boldsymbol{Y},t)\|_{r\times s}\le k(t)\|\boldsymbol{X}-\boldsymbol{Y}\|_{r\times s}$, $\int_{t_0}^{t_e}k(t)dt<\infty$.

Theorem4.1.

Under the hypothesis **C1** and **C2**, random matrix Euler method (1.32) is m.s. convergent.

Proof

If $\boldsymbol{X}(\boldsymbol{t})$ is the theoretical solution of matrix problem (1.1) then each of its entries function $X^{ij}(t)$ are m.s. differentiable and applying the m.s. fundamental theorem of calculus in the interval $[t_n,t_{n+1}]\subset[t_0,t_e]$, and considering the (i,j)-th entry of $\dot{X}(u)=F(X(u),u)$, it follows:

$$X^i(t_{n+1})-X^i(t_n)=\int_{t_n}^{t_{n+1}}\dot{X}^i(u)du=\int_{t_n}^{t_{n+1}}F^i(\boldsymbol{X}(u),u)du \qquad (1.34)$$

Taking into account the (i,j)-th entry of (1.32) and example (2.1.) with $Y=F^{ij}(\boldsymbol{X}_n,t_n)\in L_2$ one gets:

$$X_{n+1}^{ij}-X_n^{ij}=hF^{ij}(\boldsymbol{X}_n,t_n)=\int_{t_n}^{t_{n+1}}F^{ij}(\boldsymbol{X}_n,t_n)du. \qquad (1.35)$$

From (1.33)-(1.35) it follows:

$$e_{n+1}^{ij}=e_n^{ij}+\int_{t_n}^{t_{n+1}}[F^{ij}(\boldsymbol{X}_n,t_n)-F^i(\boldsymbol{X}(u),u)]du. \qquad (1.36)$$

In order to prove that the m.s. convergence of the error random matrix sequence $\boldsymbol{e}_n=(e_n^{ij})$ to the zero matrix in $L_2^{r\times s}$, as **F**(**X**, t) is m.s. continuous, by proposition (2.1.) it follows that

$$\max_{1\le i\le r}\sum_{j=1}^{s}\left\|\int_{t_n}^{t_{n+1}}[F^{ij}(\boldsymbol{X}_n,t_n)-F^{ij}(\boldsymbol{X}(u),u)]du\right\| \le \max_{1\le i\le r}\sum_{j=1}^{s}\int_{t_n}^{t_{n+1}}\|[F^{ij}(\boldsymbol{X}_n,t_n)-F^{ij}(\boldsymbol{X}(u),u)]\|du \qquad (1.37)$$

$$\le\int_{t_n}^{t_{n+1}}\max_{1\le i\le r}\sum_{j=1}^{s}\|[F^{ij}(\boldsymbol{X}_n,t_n)-F^{ij}(\boldsymbol{X}(u),u)]\|du.$$

By applying the inequality (3.10) in [8] to each component (i,j) of **F**(**X**, t) for $u\in[t_n,t_{n+1}]$, we have:

$$\|[F^{i\times j}(\boldsymbol{X}_n,t_n)-F^{i\times j}(\boldsymbol{X}(u),u)]\| \le\|[F^{i\times j}(\boldsymbol{X}_n,t_n)-F^{i\times j}(\boldsymbol{X}(t_n),t_n)]\|+\|[F^{i\times j}(\boldsymbol{X}(t_n),t_n)-F^{i\times j}(\boldsymbol{X}(u),t_n)]\|+\|[F^{i\times j}(\boldsymbol{X}(u),t_n)-F^{i\times j}(\boldsymbol{X}(u),u)]\| \qquad (1.38)$$

Note that using Lipschitz condition given by **C2** and the definition of $\|\cdot\|_{r\times s}$, we have:

$$\|F^{ij}(\boldsymbol{X}_n,t_n)-F^{ij}(\boldsymbol{X}(t_n),t_n)\| \le\|F(\boldsymbol{X}_n,t_n)-F(\boldsymbol{X}(t_n),t_n)\|_{r\times s} \le k(t_n)\|\boldsymbol{X}_n-\boldsymbol{X}(t_n)\|_{r\times s}=k(t_n)\|\boldsymbol{e}_n\|_{r\times s} =k(t_n)\|\boldsymbol{e}_n\|_{r\times s},$$

And we have:

$$\|[F^{ij}(\boldsymbol{X}(u),t_n)-F^{ij}(\boldsymbol{X}(u),u)]\| \le\|\boldsymbol{F}(\boldsymbol{X}(u),t_n)-\boldsymbol{F}(\boldsymbol{X}(u),u)\|_{r\times s} \le\omega(S_X,h), \qquad (1.40)$$

where S_X be the bounded set in $L_2^{r\times s}$ defined by the exact theoretical solution $\boldsymbol{X}(t)$ of problem (1.1) and where $S_X=\{\boldsymbol{X}(t),t_0\le t\le t_e\}$.for $\|[F^{ij}(\boldsymbol{X}(t_n),t_n)-F^{ij}(\boldsymbol{X}(u),t_n)]\|$, first note that: $\boldsymbol{F}(\boldsymbol{X}(t_n),t_n)$ and $\boldsymbol{F}(\boldsymbol{X}(u),t_n)$ depend on $(r\times s)+1$ arguments, and we shall express the above difference as the sum of $r\times s$ terms where in each term only one argument changes and all the remaining entries keep constant.

Hence, let us write:

$\boldsymbol{F}(\boldsymbol{X}(t_n), t_n) =$
$F^{ij}(X^{11}(t_n), \dots\dots, X^{1s}(t_n); \dots; X^{r1}(t_n), \dots, X^{rs}(t_n); t_n),$
$\boldsymbol{F}(\boldsymbol{X}(u), t_n)$
$= F^{ij}(X^{11}(u), \dots\dots, X^{1s}(u); \dots; X^{r1}(u), \dots, X^{rs}(u); t_n),$
and by decomposing the difference $F^{ij}(\boldsymbol{X}(t_n), t_n) - F^{ij}(\boldsymbol{X}(u), t_n)$ in $r \times s$ terms and applying the Lipschitz condition we have

$\|[F^{ij}(\boldsymbol{X}(t_n), t_n) - F^{ij}(\boldsymbol{X}(u), t_n)]\| \le rsk(t_n) \max_{1 \le i \le r} \sum_{j=1}^{s} \|X^{ij}(u) - X^{ij}(t_n)\|,$ (1.41) Because:

$$\|[F^{ij}(\boldsymbol{X}(t_n), t_n) - F^{ij}(\boldsymbol{X}(u), t_n)]\| \le \|\boldsymbol{F}(\boldsymbol{X}(t_n), t_n) - \boldsymbol{F}(\boldsymbol{X}(u), t_n)\| \le$$
$$\|\boldsymbol{F}(X^{11}(u), X^{12}(u), \dots\dots, X^{1s}(u); \dots; X^{r1}(u), \dots, X^{rs-1}(u), X^{rs}(t_n), t_n) - \boldsymbol{F}(X^{11}(u), X^{12}(u), \dots\dots, X^{1s}(u); \dots; X^{r1}(u), X^{r2}(u), \dots, X^{rs-1}(u), X^{rs}(u), t_n)\| \le$$
$$rsk(t_n)\|\boldsymbol{X}(u) - \boldsymbol{X}(t_n)\|_{r\times s} = rsk(t_n) \max_{1 \le i \le r} \sum_{j=1}^{s} \|X^{ij}(u) - X^{ij}(t_n)\|$$

As each entry $X^{ij}(t)$ of the solution $\boldsymbol{X}(t)$of problem (1.1) is m.s. differentiable, in an analogous way to (3.13)-(3.14) as in [8] it follows that:

$$\|X^{ij}(u) - X^{ij}(t_n)\| = \left\| \int_{t_n}^{u} X^{ij}(v)dv \right\| \le \int_{t_n}^{u} \|X^{ij}(v)\|dv \le hM_{\dot{X}}, \quad (1.42)$$

Where $\boldsymbol{X}(t)$ is the theoretical solution of problem (1.1) and we have applied that:

$$\|X^{ij}(v)\| \le \|\dot{X}(v)\|_{r\times s} \le M_{\dot{X}} = sup\left\{\|\dot{X}(t)\|_{r\times s}:\ t_0 \le t \le t_e\right\} \quad (1.43)$$

By subtracting (1.43) from (1.43) one get:

$$\|[F^{ij}(\boldsymbol{X}(t_n), t_n) - F^{ij}(\boldsymbol{X}(u), t_n)]\| \le s^2 rk(t_n)hM_{\dot{X}},$$

For $1 \le i \le r 1 \le j \le s$ (1.44)

From equations (1.38), (1.39) and (1.44) we have:

$\|[F^{ij}(\boldsymbol{X}_n, t_n) - F^{ij}(\boldsymbol{X}(u), u)]\| \le k(t_n)[\|\boldsymbol{e}_n\|_{r\times s} + s^2 rhM_{\dot{X}}] + \omega(S_{\boldsymbol{X}}, h)$ (1.45)

Then:

$\max_{1 \le i \le r} \sum_{j=1}^{s} \|[F^{ij}(\boldsymbol{X}_n, t_n) - F^{ij}(\boldsymbol{X}(u), u)]\| \le sk(t_n)[\|\boldsymbol{e}_n\|_{r\times s} + s^2 rhM_{\dot{X}}] + s\omega(S_{\boldsymbol{X}}, h)$ (1.46)

Taking norms in (1.136) and using (1.37), (1.46) one gets:

$$\|e_{n+1}^{ij}\| \le \|e_n^{ij}\| + \left\| \int_{t_n}^{t_{n+1}} \|[F^{ij}(\boldsymbol{X}_n, t_n) - F^{ij}(\boldsymbol{X}(u), u)]\| du \right\|$$
$$\le \|e_n^{ij}\| + \left\| \int_{t_n}^{t_{n+1}} \|[F(\boldsymbol{X}_n, t_n) - F(\boldsymbol{X}(u), u)]\| du \right\|_{r\times s}$$
$$\le \|e_n^{ij}\| + hsk(t_n)\|\boldsymbol{e}_n\|_{r\times s} + r\,h^2\,s^3 k(t_n)M_{\dot{X}} + hs\omega(S_{\boldsymbol{X}}, h).$$

Hence,

$$\|\boldsymbol{e}_{n+1}\|_{r\times s} \le \|\boldsymbol{e}_n\|_{r\times s} + h\,s^2 k(t_n)\|\boldsymbol{e}_n\|_{r\times s} + r\,h^2\,s^4 k(t_n)M_{\dot{X}} + h\,s^2\omega(S_{\boldsymbol{X}}, h)$$
$$= (1 + h\,s^2 k(t_n))\|\boldsymbol{e}_n\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)].$$

Then by substituting in (1.38) as shown in [10] we have:

$$\|\boldsymbol{e}_{n+1}\|_{r\times s} \le \|\boldsymbol{e}_n\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)\|\boldsymbol{e}_n\|_{r\times s}]$$
$$= (1 + h\,s^2 k(t_n))\|\boldsymbol{e}_n\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)]$$
$$\le (1 + h\,s^2 k(t_n))(1 + h\,s^2 k(t_n))\|\boldsymbol{e}_n\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)] + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)]$$
$$= (1 + h\,s^2 k(t_n))^2\|\boldsymbol{e}_{n-1}\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)][(1 + h\,s^2 k(t_n))] \le$$
$$(1 + h\,s^2 k(t_n))^3\|\boldsymbol{e}_{n-2}\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)]\left[1 + (1 + h\,s^2 k(t_n)) + (1 + h\,s^2 k(t_n))^2\right] \le$$
$$(1 + h\,s^2 k(t_n))^{n+1}\|\boldsymbol{e}_0\|_{r\times s} + h[rh\,s^4 k(t_n)M_{\dot{X}} + s^2\omega(S_{\boldsymbol{X}}, h)][1 + (1 + h\,s^2 k(t_n)) + (1 + h\,s^2 k(t_n))^2 + \dots\dots + (1 + h\,s^2 k(t_n))^n]$$

Since $[1 + (1 + h\,s^2 k(t_n)) + (1 + h\,s^2 k(t_n))^2 + \dots\dots + (1 + h\,s^2 k(t_n))^n]$ is geometrical series then,

$$[1 + (1 + h\,s^2 k(t_n)) + (1 + h\,s^2 k(t_n))^2 + \dots\dots + (1 + h\,s^2 k(t_n))^n] = \frac{(1 + h\,s^2 k(t_n))^n - 1}{h\,s^2 k(t_n)}$$

Hence:

$$\|\boldsymbol{e}_{n+1}\|_{r\times s} \le (1 + h\,s^2 k(t_n))^{n+1}\|\boldsymbol{e}_0\|_{r\times s} + [rh\,s^2 k(t_n)M_{\dot{X}} + \omega(S_{\boldsymbol{X}}, h)]\frac{[(1 + h\,s^2 k(t_n))^n - 1]}{k(t_n)}$$

At $h \to 0$, we have: $\|\boldsymbol{e}_{n+1}\|_{r\times s} \le$

$$e^{nhs^2 k(t_n)}\|\boldsymbol{e}_0\|_{r\times s} + \frac{e^{nhs^2 k(t_n)} - 1}{k(t_n)}[rh\,s^2 k(t_n)M_{\dot{X}} + \omega(S_{\boldsymbol{X}}, h)]$$

As $\boldsymbol{e}_0 = \boldsymbol{0}$ *and* $nh = t_n - t_0 = t - t_0$, the last expression can be written in the form:

$$\|\boldsymbol{e}_{n+1}\|_{r\times s} \le \frac{e^{n(t-t_0)s^2 k(t_n)} - 1}{k(t_n)}[rh\,s^2 k(t_n)M_{\dot{X}} + \omega(S_{\boldsymbol{X}}, h)] \quad (1.47)$$

Finally we find:

$$\|\boldsymbol{e}_{n+1}\|_{r\times s} \to 0 \ as\ h \to 0, n \to \infty, nh = t - t_0.$$

Then random matrix Euler method (1.32) is convergent in mean square sense.

V. CASES STUDY

Example:

Consider the linear system of two random differential equations:

$$\begin{cases} x'(t) = \alpha x + \beta y & x(0) = 2 \\ y'(t) = \beta x + \alpha y & y(0) = 0, \end{cases}$$
$$where\ \ t \in [0, t_n]$$

Where α, β are independent random variables with joint probability density function of the uniform distribution:

$$f(\alpha, \beta) = 1, 0 \le \alpha \le 1\ and\ 0 \le \beta \le 1.$$

The exact solution:

$$\begin{cases} x(t) = e^{(\alpha+\beta)t} + e^{(\alpha-\beta)t} \\ y(t) = e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t} \end{cases}$$

Or the exact solution is the system:

$$X(t) = \begin{bmatrix} e^{(\alpha+\beta)t} + e^{(\alpha-\beta)t} \\ e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t} \end{bmatrix}$$

Where: $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

The numerical solution:

By using random matrix Euler method:

$$\boldsymbol{X}_{n+1} = \boldsymbol{X}_n + h\boldsymbol{F}(\boldsymbol{X}_n, t_n)$$

Then: $x_{n+1}(t) = x_n(t) + h(\alpha x_n(t) + \beta y_n(t)),$

$y_{n+1}(t) = y_n(t) + h(\beta x_n(t) + \alpha y_n(t))$

With:

$$x(0) = 2 \ and \ y(0) = 0$$

Then we have:

$$\begin{cases} x_{n+1}(t) = (1+\alpha h)x_n(t) + \beta y_n(t) & x(0) = 2 \\ y_{n+1}(t) = \beta h x_n(t) + (1+\alpha h) y_n(t) & y(0) = 0 \end{cases}$$

This system can be written as:

$$X_{n+1}(t) = AX_n(t) \qquad (*)$$

Where:

$$A = \begin{bmatrix} 1+\alpha h & \beta h \\ \beta h & 1+\alpha h \end{bmatrix} and \ \ X_n(t) = \begin{bmatrix} x_n(t) \\ y_n(t) \end{bmatrix}$$

We will find the eigenvalues (λ_1, λ_2) of the matrix A,

Since:

$$|A - \lambda I| = 0$$

Then:

$$\begin{bmatrix} 1+\alpha h - \lambda & \beta h \\ \beta h & 1+\alpha h - \lambda \end{bmatrix} = 0$$

Then:

$$\lambda_1 = 1 + \alpha h + \beta h$$
$$\lambda_1 = 1 + \alpha h - \beta h$$

Then the solutions are:

$$x_n(t) = C_1\lambda_1^n + C_2\lambda_2^n$$

At: $x(0) = 2 \quad then \quad C_1 + C_2 = 2$

if $we \ take \ C_1 = C_2 = 1$ then:

$$x_n(t) = (1+\alpha h+\beta h)^n + (1+\alpha h-\beta h)^n$$

And, $\quad y_n(t) = C_1\lambda_1^n + C_2\lambda_2^n$

At $y(0) = 0 \quad then \quad C_1 + C_2 = 0$

if $we \ take \ C_1 = 1 \ and \ C_2 = -1$ then:

$$y_n(t) = (1+\alpha h+\beta h)^n - (1+\alpha h-\beta h)^n$$

Then the numerical solutions of the system are:

$$\begin{cases} x_n(t) = (1+\alpha h+\beta h)^n + (1+\alpha h-\beta h)^n \\ y_n(t) = (1+\alpha h+\beta h)^n - (1+\alpha h-\beta h)^n \end{cases}$$

Or the numerical solution of the system (*) is

$$X_n(t) = \begin{bmatrix} (1+\alpha h+\beta h)^n + (1+\alpha h-\beta h)^n \\ (1+\alpha h+\beta h)^n - (1+\alpha h-\beta h)^n \end{bmatrix}$$

Verification:

$$(i) \lim_{n\to\infty} X_n(t) = X(t)$$

We will prove that:

$$\lim_{n\to\infty} x_n(t) = x(t)$$

And:

$$\lim_{n\to\infty} y_n(t) = y(t)$$

First:

$$\lim_{n\to\infty} x_n(t) = x(t)$$

Since:

$$x_n(t) = (1+\alpha h+\beta h)^n + (1+\alpha h-\beta h)^n$$
$$= \left(1+h(\alpha+\beta)\right)^n + \left(1+h(\alpha-\beta)\right)^n$$

Then:

$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\{\left(1+h(\alpha+\beta)\right)^n + \left(1+h(\alpha-\beta)\right)^n\} = \lim_{n\to\infty}\{\left(1+h(\alpha+\beta)\right)^n\} + \lim_{n\to\infty}\{\left(1+h(\alpha-\beta)\right)^n\}$

Where: $\quad h = \frac{t_n - t_0}{n} = \frac{t_n - 0}{n} = \frac{t_n}{n}$

Then:

$$\lim_{n\to\infty} x_n(t) = \lim_{n\to\infty}\{\left(1+\frac{t_n}{n}(\alpha+\beta)\right)^n\} + \lim_{n\to\infty}\{\left(1+\frac{t_n}{n}(\alpha-\beta)\right)^n\}$$
$$= e^{(\alpha+\beta)t} + e^{(\alpha-\beta)t} = x(t)$$

Second:

$$\lim_{n\to\infty} y_n(t) = y(t)$$

Since:

$$y_n(t) = (1+\alpha h+\beta h)^n - (1+\alpha h-\beta h)^n$$
$$= \left(1+h(\alpha+\beta)\right)^n - \left(1+h(\alpha-\beta)\right)^n$$

Then:

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty}\{\left(1+h(\alpha+\beta)\right)^n - \left(1+h(\alpha-\beta)\right)^n\}$$
$$= \lim_{n\to\infty}\{\left(1+h(\alpha+\beta)\right)^n\} - \lim_{n\to\infty}\{\left(1+h(\alpha-\beta)\right)^n\}$$

Where : $\quad h = \frac{t_n - t_0}{n} = \frac{t_n - 0}{n} = \frac{t_n}{n}$

Then:

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty}\{\left(1+\frac{t_n}{n}(\alpha+\beta)\right)^n\} - \lim_{n\to\infty}\{\left(1+\frac{t_n}{n}(\alpha-\beta)\right)^n\}$$
$$= e^{(\alpha+\beta)t} - e^{(\alpha-\beta)t} = y(t)$$

Hence:

$$\lim_{n\to\infty} X_n(t) = X(t)$$

Now,

$$(ii) \lim_{n\to\infty} E\{X_n(t)\} = E\{X(t)\}$$

We will prove that:

$$\lim_{n\to\infty} E\{x_n(t)\} = E\{x(t)\}$$

And

$$\lim_{n\to\infty} E\{y_n(t)\} = E\{y(t)\}$$

First:

$$\lim_{n\to\infty} E\{x_n(t)\} = E\{x(t)\}$$

Since:

$$x_n(t) = (1+\alpha h+\beta h)^n + (1+\alpha h-\beta h)^n$$
$$= \left(1+h(\alpha+\beta)\right)^n + \left(1+h(\alpha-\beta)\right)^n$$
$$= \sum_{k=0}^{n} \binom{n}{k} h^k\{(\alpha+\beta)^k + (\alpha-\beta)^k\}$$

Then:

$$E\{x_n(t)\} = E\left\{\sum_{k=0}^{n} \binom{n}{k} h^k\{(\alpha+\beta)^k + (\alpha-\beta)^k\}\right\}$$
$$= \sum_{k=0}^{n} \binom{n}{k} h^k E\left\{\{(\alpha+\beta)^k + (\alpha-\beta)^k\}\right\}$$

We will find $E\left\{\{(\alpha+\beta)^k + (\alpha-\beta)^k\}\right\}$ since $\alpha, \beta \sim U(0,1) \ and$ joint $PDF \ f(\alpha, \beta) = 1 \ then$

$$E\left\{\{(\alpha+\beta)^k+(\alpha-\beta)^k\}\right\}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\alpha+\beta)^k+(\alpha-\beta)^k\}\,f(\alpha,\beta)\,d\beta d\alpha$$
$$=\int_0^1\int_0^1\{(\alpha+\beta)^k+(\alpha-\beta)^k\}\,d\beta d\alpha$$
$$=\int_0^1\left\{\left\{\frac{(\alpha+\beta)^{k+1}}{k+1}\right\}-\left\{\frac{(\alpha-\beta)^{k+1}}{k+1}\right\}\right\}\Big|_0^1 d\alpha$$
$$=\int_0^1\left\{\frac{(\alpha+1)^{k+1}}{k+1}\right\}-\left\{\frac{(\alpha-1)^{k+1}}{k+1}\right\}d\alpha$$
$$=\left\{\left\{\frac{(\alpha+1)^{k+2}}{(k+1)(k+2)}\right\}-\left\{\frac{(\alpha-1)^{k+2}}{(k+1)(k+2)}\right\}\right\}\Big|_0^1$$
$$=\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}$$

Then:

$$E\{x_n(t)\}=\sum_{k=0}^{n}\binom{n}{k}h^k\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}$$

Hence:

$$\lim_{n\to\infty}E\{x_n(t)\}=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}h^k\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}$$
$$=\sum_{k=0}^{\infty}\lim_{n\to\infty}\{\binom{n}{k}h^k\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}\}$$

Where: $h=\frac{t_n-t_0}{n}=\frac{t_n-0}{n}=\frac{t_n}{n}$

$$\lim_{n\to\infty}E\{x_n(t)\}$$
$$=\sum_{k=0}^{\infty}\lim_{n\to\infty}\{\frac{n!}{k!\,(n-k)!}\left(\frac{t_n}{n}\right)^k\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}$$

Since:

$$x(t)=e^{(\alpha+\beta)t}+e^{(\alpha-\beta)t}$$

Then:

$$x(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha+\beta)^k+\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha-\beta)^k$$

Then:

$$E\{x(t)\}=E\left\{\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha+\beta)^k+\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha-\beta)^k\right\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}E\{(\alpha+\beta)^k+(\alpha-\beta)^k\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{(2)^{k+2}-1+(-1)^{k+2}}{(k+1)(k+2)}=\lim_{n\to\infty}E\{x_n(t)\}$$

i.e. $\lim_{n\to\infty}E\{x_n(t)\}=E\{x(t)\}$

Second:

$$\lim_{n\to\infty}E\{y_n(t)\}=E\{y(t)\}$$

Since:

$$y_n(t)=(1+\alpha h+\beta h)^n-(1+\alpha h-\beta h)^n$$
$$=\left(1+h(\alpha+\beta)\right)^n-\left(1+h(\alpha-\beta)\right)^n$$
$$=\sum_{k=0}^{n}\binom{n}{k}h^k\{(\alpha+\beta)^k-(\alpha-\beta)^k\}$$

Then:

$$E\{y_n(t)\}=E\left\{\sum_{k=0}^{n}\binom{n}{k}h^k\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\right\}$$
$$=\sum_{k=0}^{n}\binom{n}{k}h^kE\left\{\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\right\}$$

We will find $E\left\{\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\right\}$ since, $\alpha,\beta\sim U(0,1)$ and joint PDF $f(\alpha,\beta)=1$ then:

$$E\left\{\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\right\}$$
$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\,f(\alpha,\beta)\,d\beta d\alpha$$
$$=\int_0^1\int_0^1\{(\alpha+\beta)^k-(\alpha-\beta)^k\}\,d\beta d\alpha$$
$$=\int_0^1\left\{\left\{\frac{(\alpha+\beta)^{k+1}}{k+1}\right\}-\left\{\frac{(\alpha-\beta)^{k+1}}{k+1}\right\}\right\}\Big|_0^1 d\alpha$$
$$=\int_0^1\left\{\frac{(\alpha+1)^{k+1}}{k+1}\right\}+\left\{\frac{(\alpha-1)^{k+1}}{k+1}\right\}d\alpha$$
$$=\int_0^1\left\{\frac{(\alpha+1)^{k+1}}{k+1}\right\}+\left\{\frac{(\alpha-1)^{k+1}}{k+1}\right\}d\alpha$$
$$=\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}$$

Then:

$$E\{y_n(t)\}=\sum_{k=0}^{n}\binom{n}{k}h^k\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}$$

Hence:

$$\lim_{n\to\infty}E\{y_n(t)\}=\lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}h^k\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}$$
$$=\sum_{k=0}^{\infty}\lim_{n\to\infty}\{\binom{n}{k}h^k\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}\}$$

Where: $h=\frac{t_n-t_0}{n}=\frac{t_n-0}{n}=\frac{t_n}{n}$

Then:

$$\lim_{n\to\infty}E\{y_n(t)\}$$
$$=\sum_{k=0}^{\infty}\lim_{n\to\infty}\{\frac{n!}{k!\,(n-k)!}\left(\frac{t_n}{n}\right)^k\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}$$

Since:

$$y(t)=e^{(\alpha+\beta)t}-e^{(\alpha-\beta)t}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha+\beta)^k-\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha-\beta)^k$$

Then:

$$E\{y(t)\}=E\left\{\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha+\beta)^k-\sum_{k=0}^{\infty}\frac{t^k}{k!}(\alpha-\beta)^k\right\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}E\{(\alpha+\beta)^k-(\alpha-\beta)^k\}$$
$$=\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{(2)^{k+2}-1-(-1)^{k+2}}{(k+1)(k+2)}=\lim_{n\to\infty}E\{y_n(t)\}$$

i.e. $(ii)\lim_{n\to\infty}E\{y_n(t)\}=E\{y(t)\}$

Then:

$$\lim_{n\to\infty}E\{X_n(t)\}=E\{X(t)\}$$

VI. CONCLUSIONS

The initially valued system of first order random differential equations can be solved numerically using the random Euler method in mean square sense. The existence and uniqueness of the solution have been proved. The convergence of the random Euler method has been proved in mean square sense. The results of the paper have been illustrated through some examples.

REFERENCES

[1] Barron, R., Ayala, G.: El m´etodo de yuxtaposici´on de dominios en la soluci´on num´erica de ecuaciones diferenciales estoc´asticas. In: Proc. Metodos Num´ericos en Ingenier´ıa y Ciencias Aplicadas (CIMNE), Monterrey, Mexico, pp. 267–276 (2002).

[2] Braumann, C.A.: Variable effort harvesting models in random environments: generalization to density-dependent noise intensities. Math. Biosci. 177-178, 229–245 (2002)

[3] Chil`es, J., Delfiner, P.: Geostatistics. Modelling Spatial Uncertainty. John Wiley, New York (1999)

[4]Keller, J.B.: Wave propagation in random media. In: Proc. Symp. Appl. Math., vol. 13, pp. 227–246. Amer. Math. Soc., Providence (1962)

[5] Keller, J.B.: Stochastic equations and wave propagation in random media. In: Proc. Symp. Appl. Math., New York, vol. 16, pp. 145–170. Amer. Math. Soc., Providence (1964)

[6]Talay, D.: Expansion of the global error for numerical schemes solving stochastic differential equations. Stochastics Analysis and Applications 8(4), 483–509 (1990)

[7] T.T. Soong (1973). Random Differential Equations in Science and Engineering, Academic Press, New York.

[8] J.C. Cort´es, L. J´odar, R.J.Villanueva, L. Villafuerte (2010). Mean Square Convergent Numerical Mehods for Nonlinear Random Differential Equations, Comput.Math. Appl. 59, 1–21.

[9] Golub, G., Van Loan, C.F.: Matrix Computations. The Johns Hopkins University Press, Baltimore (1989).

[10] Magdy, A. El-Tawil and Mohammed A.sohaly, Mean Square Numerical Methods for initial value random differential equations, open Journal of Discrete Mathematics OJDM, 2011(1), 66–84.

[11]] M. El-Sohaly, Mean square convergent three and five points finite difference scheme for stochastic parabolic partial differential equations, Electronic Journal of Mathematical Analysis and Applications 2 (2014) 66–84

